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SHAKEDOWN LOADS FOR RADIAL NOZZLES IN SPHERICAL PRESSURE VESSELS

F. A. LECKIE and R. K. PENNY

Department of Engineering, University of Cambridge

Abstract—Lower bound estimates have been found of shakedown values for pressure and thrust and moment loadings applied through a radial nozzle in a spherical pressure vessel; the nozzle can be flush or protruding. These have been achieved by using Melan's theorem and by exploiting elastic solutions already available. The results were obtained by using standard linear programming techniques and have been presented in a useful graphical form.

NOTATION

H, M	edge force and bending moment respectively per unit circumference
$M_0, \bar{m}$	external nozzle moment and moment shakedown factor
$p, \bar{p}$	pressure and pressure shakedown factor
$Q, \bar{q}$	external nozzle thrust loading and thrust shakedown factor
R, r	radii of spherical and cylindrical shells
T, t	thickness of spherical and cylindrical shells
$\alpha, \beta, \gamma, \delta$	residual stress group factors
θ, ϕ	angles of longitude and latitude in spherical shell
ρ	geometrical factor for spherical shell $\left[= \frac{r}{R}\sqrt{\left(\frac{R}{T}\right)}\right]$
σ	normal stress
σ^*	yield stress in simple tension

INTRODUCTION

In present day designs the stresses which occur at the junction of nozzles in steel pressure vessels are normally high enough to cause plastic deformations. While it is possible to calculate the stress and strain distribution in the elastic–plastic range, the results of these calculations are of limited use because they are so dependent on residual stresses and loading history. Ultimate load and shakedown performance however are history independent and for that reason do provide real guidance in making design decisions. When a structure is subjected to static loading then a knowledge of the ultimate load is usually sufficient. However, when the loading is cyclic, shakedown performance becomes important. If the cyclic loading is kept within the shakedown limit then the designer is assured that, after initial plastic deformation, further deformation is in the elastic range, and that the possibilities of incremental collapse or reversed plasticity are removed.

Ultimate and some lower bound estimates of shakedown pressure have already been obtained for the radial nozzle [1–3] intersecting a spherical pressure vessel (Fig. 1). In [3] it was shown that the lower bound shakedown estimates can be obtained by using Melan's Shakedown Theorem [4] in conjunction with the results of a previously computed elastic analysis. This procedure is repeated in the present paper but this time other loadings

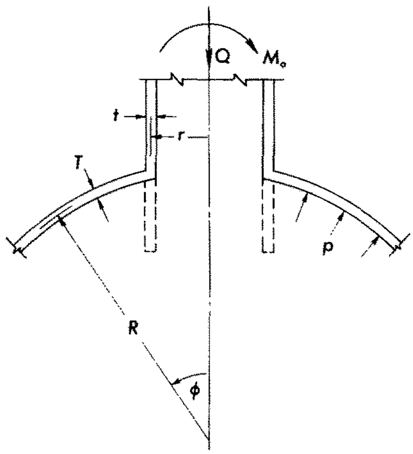

FIG. 1. Shell geometry and loading.

that are liable to occur during the operation of the pressure vessel are considered. These are the cases of a thrust and a moment applied to the nozzle; in addition the flush and the protruding nozzles are considered.

Finally, since it is very likely in practice that the above loadings will occur simultaneously, it is of considerable interest to determine to what extent these interact. In the paper this problem is considered in detail and interaction surfaces for the shakedown loads are generated.

ASSUMPTIONS AND DEFINITIONS

In practical pressure vessels it is found [5] that the maximum stresses occur in the spherical portion at the point of intersection, although in some exceptional geometries, when the nozzle is very thin [6], or the opening is large, maximum stresses can occur in the nozzle. Such exceptional cases are excluded here, and the investigation is confined to a study of stresses in the sphere; this is not a necessary restriction, however. The material of the shell is assumed to be elastic/perfectly plastic, and to yield according to the Tresca yield criterion. Accordingly if the stresses at the surface are σ_ϕ in the meridional direction, σ_θ in the circumferential direction, and if the radial stress is neglected then the yield criteria are

$$|\sigma_\phi| \leq \sigma^*, \qquad |\sigma_\theta| \leq \sigma^*, \qquad |\sigma_\phi - \sigma_\theta| \leq \sigma^* \tag{1}$$

where σ^* is the yield stress obtained from a simple tension test.

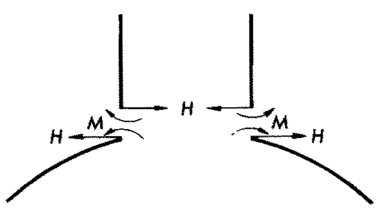

FIG. 2. Edge forces for axisymmetric loading.

THE ELASTIC SOLUTIONS

Elastic solutions are available [7] for the present shell geometries subjected to pressure, thrust and moment loadings. The solutions were obtained using the usual procedure of shell analysis of superimposing on to the membrane solution the effects of the edge forces which ensure compatibility of displacements of adjoining shells at their junction. In the case of the axisymmetric loadings of pressure and thrust the two self-equilibrating edge forces are the horizontal force H and the moment M (Fig. 2). In the case of the moment loading two independent self-equilibrating edge force groups are also available [7]. These edge groups are illustrated in Fig. 3, the forces varying around the circumference according to the sine or cosine of the longitudinal angle θ.

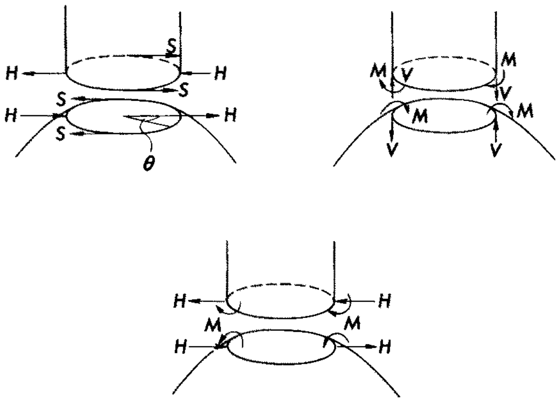

FIG. 3. Edge forces for moment loading.

THE SHAKEDOWN CALCULATION FOR PRESSURE THRUST AND MOMENT APPLIED SEPARATELY

The proposed calculations are based on Melan's theorem [4] which states: if any distribution of self-equilibrating residual stresses can be found which, when taken together with the "elastic" stresses (i.e. assuming perfectly elastic behaviour) constitute a system of stresses within the yield limit, then the structure will shake down.

The purpose of the present paper is to generate suitable self-equilibrating residual stress distributions which, in themselves, do not violate the yield conditions. The residual stress distributions are optimized to give the highest load such that the total of the elastic stresses due to the loads and the residual stresses do not exceed yield. The shakedown loads so determined are always less than the actual values since the structure automatically finds the best residual stress distribution, a circumstance which is very unlikely with assumed stress distribution.

In the case of the shell structure under consideration, the local increases of stress are due to the edge forces H and M, and in order to counteract their effect it would seem reasonable to postulate residual self-equilibrating edge forces $\bar{H}$ and $\bar{M}$ in directions opposite to those of H and M. Within the shell itself, suitable stress systems in equilibrium with $\bar{H}$ and $\bar{M}$ are provided by the linear elastic theory.

In the following calculations it is generally not convenient to deal with the single effects of H and M, but to consider rather, the effects of a force group (H, M) in which H and M remain in a fixed proportion.

(i) *Pressure loading*

For a geometry with given values of r/R, R/T and $t/T = (t/T)_1$ the stresses at the junction (where they are most severe) are:

	Inner surface	Outer surface
σ_ϕ/σ^*	$p_{11}\bar{p}$	$p_{12}\bar{p}$
σ_θ/σ^*	$p_{13}\bar{p}$	$p_{14}\bar{p}$

where

$$\bar{p} = \frac{pR}{2T\sigma^*}$$

The stresses caused by the edge forces (H_1, M_1) acting alone are found by subtracting the membrane stresses, leaving the following stresses:

	Inner surface	Outer surface
σ_ϕ/σ^*	$(p_{11}-1)\bar{p}$	$(p_{12}-1)\bar{p}$
σ_θ/σ^*	$(p_{13}-1)\bar{p}$	$(p_{14}-1)\bar{p}$

If residual values $\overline{H}_1$ and $\overline{M}_1$ are chosen such that $\overline{H}_1 = (-\alpha/p)H_1$ and $\overline{M}_1 = (-\alpha/p)M_1$ then the maximum residual stresses are:

	Inner surface	Outer surface	
σ_ϕ/σ^*	$-(p_{11}-1)\alpha$	$-(p_{12}-1)\alpha$	(2)
σ_θ/σ^*	$-(p_{13}-1)\alpha$	$-(p_{14}-1)\alpha$	

These stresses will be referred to as the α residual stress group.

A second set of residual stresses can be found by using the results of the elastic calculation for the same values r/R, R/T but using a different value for $t/T = (t/T)_2$. This ensures that the edge forces H_2 and M_2 resulting from such calculations are in a proportion different from that of the H_1, M_1 edge forces. Proceeding as for the α residual stress group and assuming this time residual values of $\overline{H}_2 = -(\beta/p)H_2$ and $\overline{M}_2 = -(\beta/p)M_2$ then the stresses resulting from the β residual stress group are:

	Inner surface	Outer surface	
σ_ϕ/σ^*	$-(p_{21}-1)\beta$	$-(p_{22}-1)\beta$	
σ_θ/σ^*	$-(p_{23}-1)\beta$	$-(p_{24}-1)\beta$	(3)

Other stress residuals determined in this way, by selecting another thickness ratio $(t/T)_3$ say, would simply be a linear combination of the previous two results since, at the level of the present calculations, the only unknowns are the residual horizontal force $\bar{H}$ and the residual moment $\bar{M}$.

Using the α and β groups as the assumed residual stresses the total elastic and residual stresses in the loaded state are:

	Inner surface	Outer surface
σ_ϕ/σ^*	$p_{11}\bar{p}-(p_{11}-1)\alpha-(p_{21}-1)\beta$	$p_{12}\bar{p}-(p_{12}-1)\alpha-(p_{22}-1)\beta$
σ_θ/σ^*	$p_{13}\bar{p}-(p_{13}-1)\alpha-(p_{23}-1)\beta$	$p_{14}\bar{p}-(p_{14}-1)\alpha-(p_{24}-1)\beta$

(4)

and in the unloaded state the stresses are given by the above stresses with $\bar{p} = 0$.

The problem is now to find the values of α and β which maximize the value of p according to the 12 limiting conditions

$$-1 \le \frac{\sigma_\phi}{\sigma^*} \le 1, \qquad -1 \le \frac{\sigma_\theta}{\sigma^*} \le 1, \qquad -1 \le \frac{\sigma_\phi - \sigma_\theta}{\sigma^*} \le 1 \tag{5}$$

for the inside and outside surfaces and in the loaded and unloaded conditions.

This problem can be classified as falling within the "standard form" of linear programming.

The standard form of linear programming can be stated as follows: maximize $f = c_j x_j$ subject to the conditions

$$b_i^L \le y_i = a_{ij}x_j \le b_i^u; \qquad b_i^L \text{ and } b_i^u$$

are the lower and upper values of a fixed vector. In the present case

$$b_i^L = \{-1\ -1\ -1\ -1\ -1\ -1\ -1\ -1\ -1\ -1\ -1\ -1\}$$

$$b_i^u = \{\ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\}$$

$$a_{ij} = \begin{bmatrix} p_{11} & -(p_{11}-1) & -(p_{21}-1) \\ p_{13} & -(p_{13}-1) & -(p_{23}-1) \\ p_{11}-p_{13} & -(p_{11}-p_{13}) & -(p_{21}-p_{23}) \\ p_{12} & -(p_{12}-1) & -(p_{22}-1) \\ p_{14} & -(p_{14}-1) & -(p_{24}-1) \\ p_{12}-p_{14} & -(p_{12}-p_{14}) & -(p_{22}-p_{24}) \\ 0 & -(p_{11}-1) & -(p_{21}-1) \\ 0 & -(p_{13}-1) & -(p_{23}-1) \\ 0 & -(p_{11}-p_{13}) & -(p_{21}-p_{23}) \\ 0 & -(p_{12}-1) & -(p_{22}-1) \\ 0 & -(p_{14}-1) & -(p_{24}-1) \\ 0 & -(p_{12}-p_{14}) & -(p_{22}-p_{24}) \end{bmatrix} \tag{6}$$

$$x_j = \{\bar{p} \quad \alpha \quad \beta\}, \qquad c_j = \{1 \quad 0 \quad 0\}$$

This process was performed on a computer for a large number of shell geometries for both flush and protruding nozzles. Plotting the shakedown pressures on the basis of the geometric parameter [7] $\rho = r/R\sqrt{(R/T)}$ yields the graphs shown in Figs. 4(a) and 4(b).

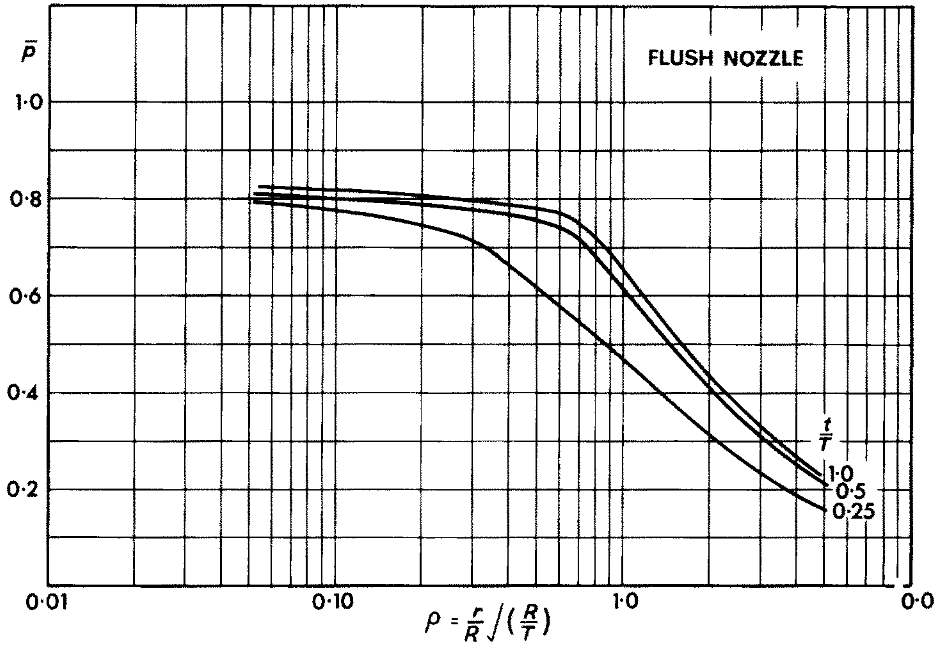

FIG. 4(a). Shakedown values for pressure loading (flush nozzle).

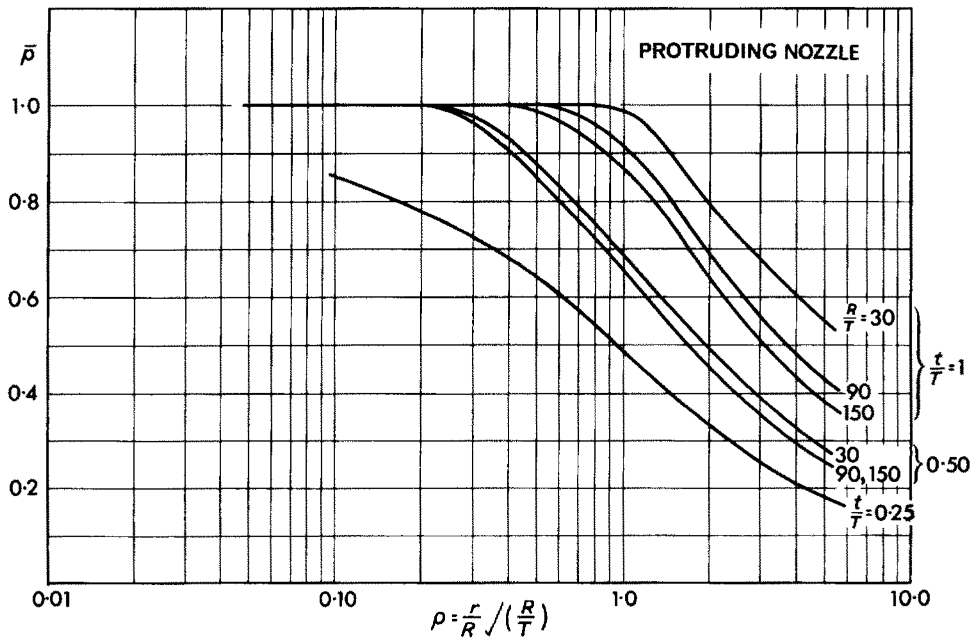

FIG. 4(b). Shakedown values for pressure loading (protruding nozzle).

(ii) *Thrust loading*

The shakedown calculations for thrust loading are similar to those for pressure loading. For the geometry r/R, R/T and $t/T = (t/T)_1$ let the maximum stresses be:

	Inner surface	Outer surface
σ_ϕ/σ^*	$q_{11}\bar{q}$	$q_{12}\bar{q}$
σ_θ/σ^*	$q_{13}\bar{q}$	$q_{14}\bar{q}$

where

$$\bar{q} = \frac{Q}{T\sigma^*}\sqrt{\frac{R}{T}}.$$

Suitable residual stresses are those resulting from the α and β groups obtained previously. The procedure follows that outlined previously, and the results obtained for the shakedown thrust loading are shown in Figs. 5–7.

(iii) *Moment loading*

When the nozzle is subjected to a moment load the meridional stress σ_ϕ and circumferential stress α_θ vary according to $\cos\theta$, while the shear stress $\sigma_{\phi\theta}$ varies according to $\sin\theta$. Experience shows that $\sigma_{\phi\theta}$ can be safely neglected by comparison with σ_ϕ and σ_θ and consequently the stress conditions at $\theta = \pi/2$ are neglected. The maximum values for

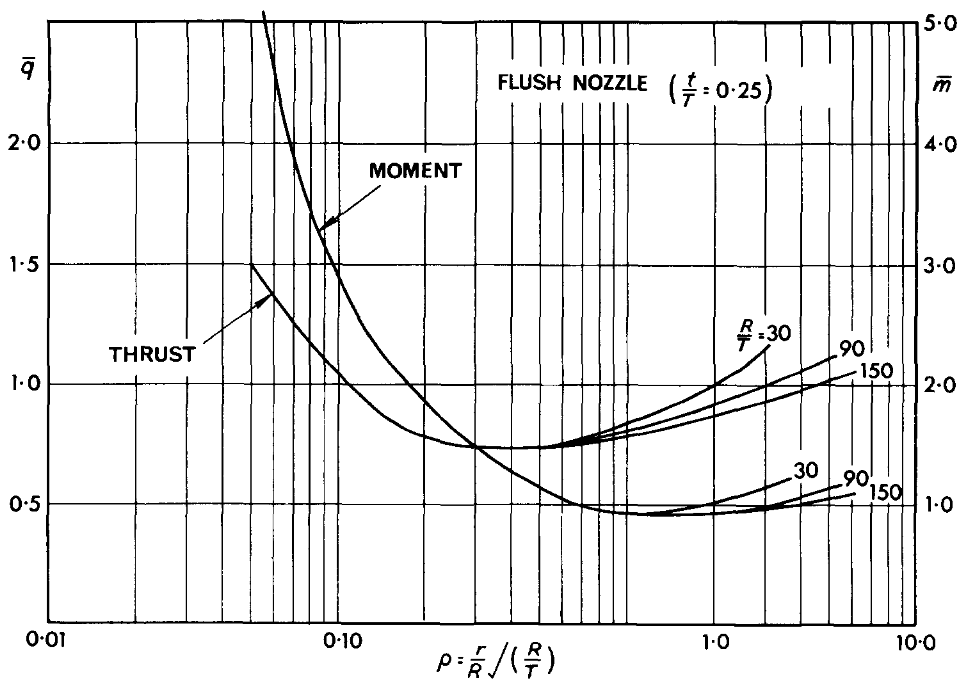

FIG. 5(a). Shakedown values for thrust and moment loadings (flush nozzle).

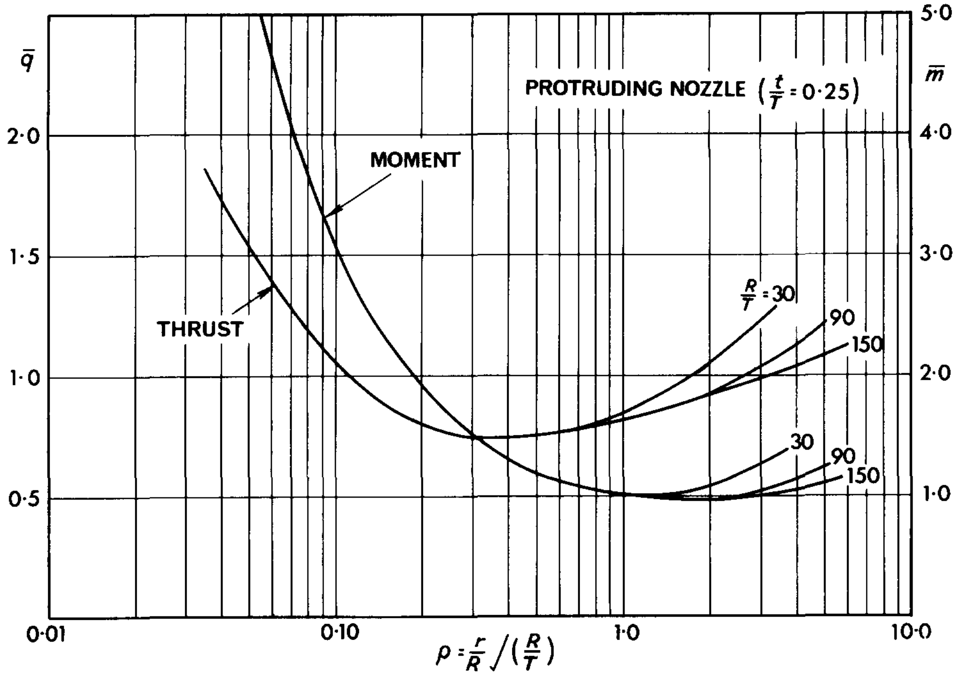

FIG. 5(b). Shakedown values for thrust and moment loadings (protruding nozzle).

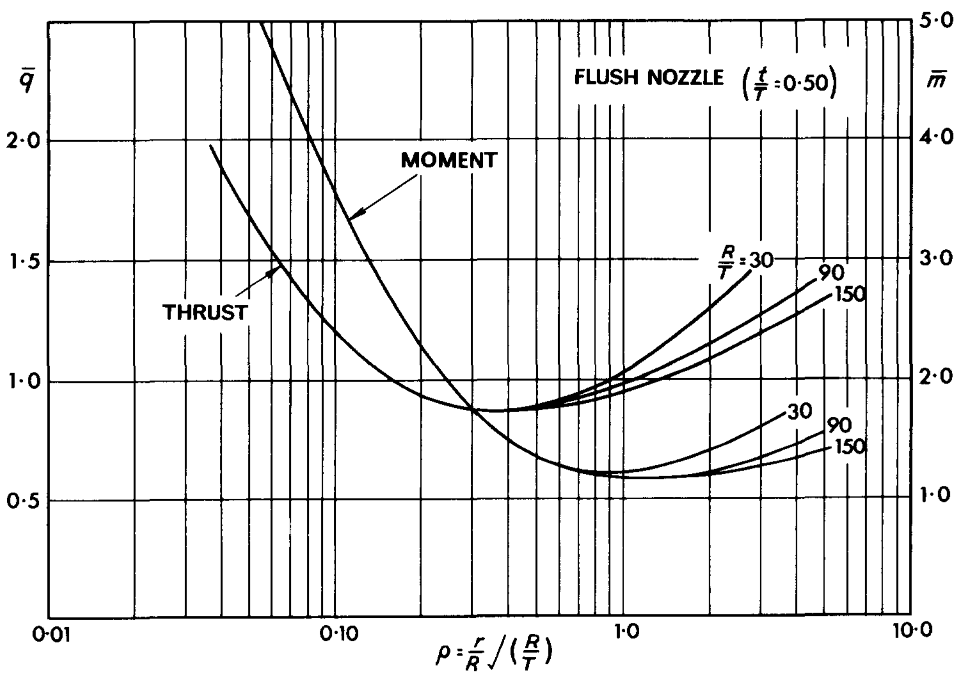

FIG. 6(a). Shakedown values for thrust and moment loadings (flush nozzles).

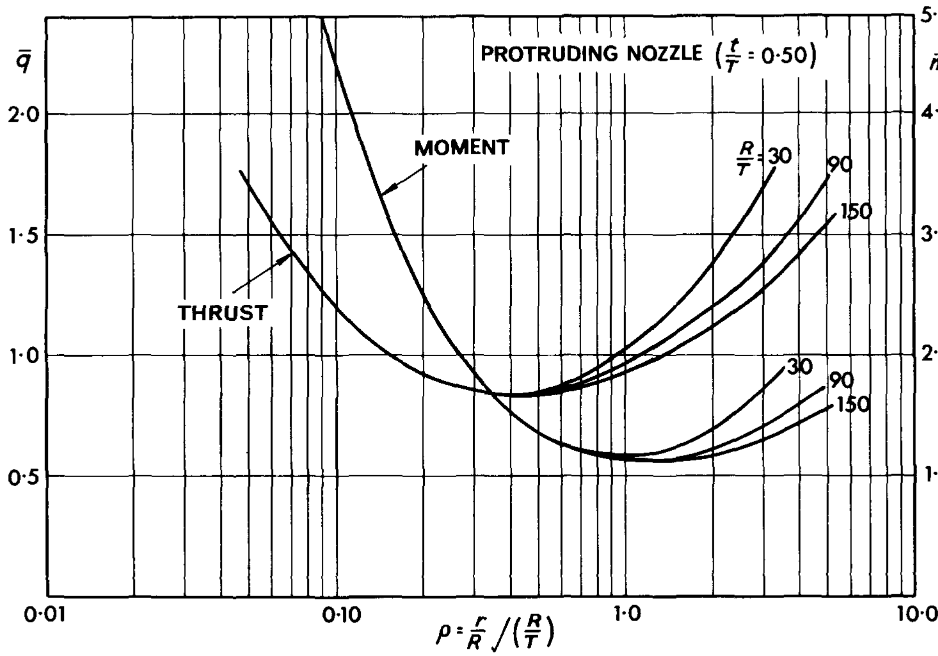

FIG. 6(b). Shakedown values for thrust and moment loadings (protruding nozzles).

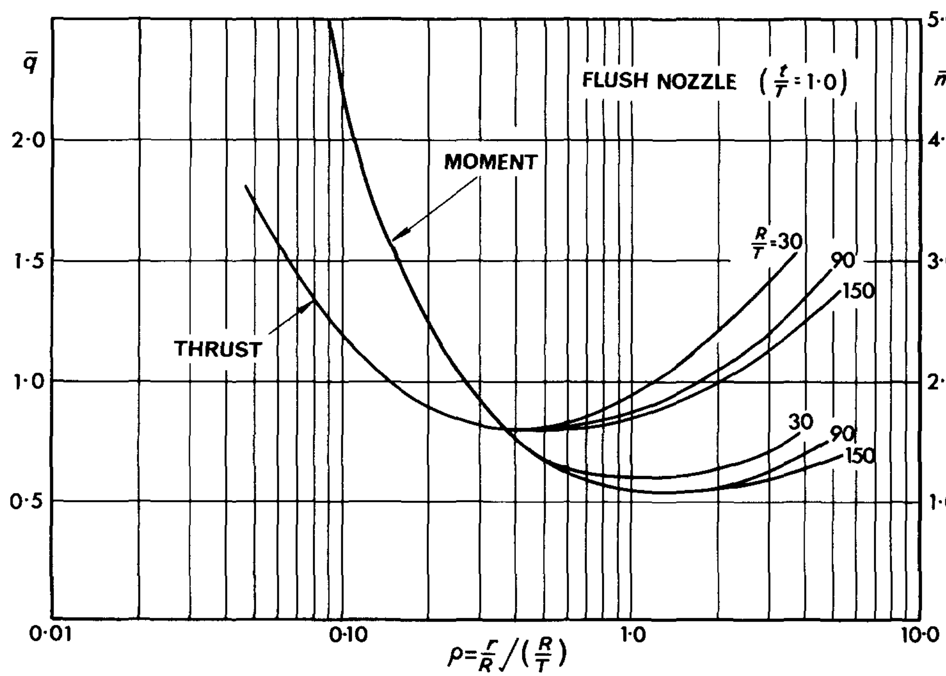

FIG. 7(a). Shakedown values for thrust and moment loadings (flush nozzle).

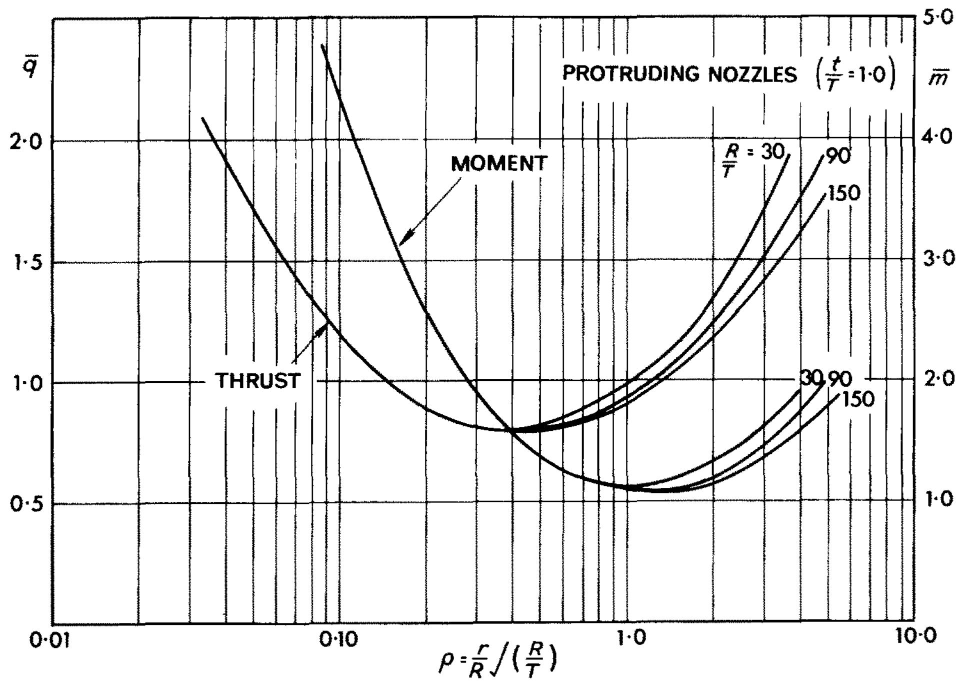

FIG. 7(b). Shakedown values for thrust and moment loadings (protruding nozzles).

σ_ϕ and σ_θ are at $\theta = 0$ and $\theta = \pi$. Since however they vary according to $\cos\theta$ only the values at either $\theta = 0$ or $\theta = \pi$ need be considered. Let the maximum elastic stresses for the geometry r/R, R/T and $t/T = (t/T)_1$ be given by

	Inner surface	Outer surface
σ_ϕ/σ^*	$m_{11}\bar{m}$	$m_{12}\bar{m}$
σ_θ/σ^*	$m_{13}\bar{m}$	$m_{14}\bar{m}$

where

$$\bar{m} = \frac{M_0}{\sigma^* \pi r^2 T}\sqrt{\frac{R}{T}}.$$

The stresses caused by the edge forces are

	Inner surface	Outer surface
σ_ϕ/σ^*	$(m_{11} - m_\phi)\bar{m}$	$(m_{12} - m_\phi)\bar{m}$
σ_θ/σ^*	$(m_{13} - m_\theta)\bar{m}$	$(m_{14} - m_\theta)\bar{m}$

where $m_\phi = -m_\theta = 1/\rho$ are the membrane stresses at the junction due to the applied moment.

Hence the γ residual stress group yields

	Inner surface	Outer surface
σ_ϕ/σ^*	$-(m_{11}-m_\phi)\gamma$	$-(m_{12}-m_\phi)\gamma$
σ_θ/σ^*	$-(m_{13}-m_\theta)\gamma$	$-(m_{14}-m_\theta)\gamma$

The δ residual stress group is determined in the same way by using the elastic solution for the geometry r/r, R/T, $(t/T)_2$, $(t/T)_2 \neq (t/T)_1$ so that

	Inner surface	Outer surface
σ_ϕ/σ^*	$-(m_{21}-m_\phi)\delta$	$-(m_{22}-m_\phi)\delta$
σ_θ/σ^*	$-(m_{23}-m_\theta)\delta$	$-(m_{24}-m_\theta)\delta$

The procedure again follows that outlined previously. The results of the shakedown loads are shown in Figs. 5–7.

INTERACTION SURFACES FOR SHAKEDOWN LOADS OF PRESSURE THRUST AND MOMENT

The aim is to obtain the shakedown interaction surface for $\bar{p}$, $\bar{q}$ and $\bar{m}$, the loads being allowed to vary arbitrarily between the limits 0 and $\bar{p}$, 0 and $\bar{q}$ and 0 and $\bar{m}$.

Because the stresses arising from pressure and thrust loadings are axisymmetric and the stresses arising from the moment loading vary as $\cos\theta$, then the stresses must be studied at $\theta = 0$, and $\theta = \pi$. In addition to the elastic stresses caused by the three loadings, residual stress groups of the α, β, γ and δ types will yield the following total stresses

	Inner surface	Outer surface
σ_ϕ/σ^*	$p_{11}\bar{p}+q_{11}\bar{q}+m_{11}\bar{m}$	$p_{12}\bar{p}+q_{12}\bar{q}+m_{12}\bar{m}$
$(\theta = 0)$	$-(p_{11}-1)\alpha-(p_{21}-1)\beta$	$-(p_{12}-1)\alpha-(p_{22}-1)\beta$
	$-(m_{11}-m_\phi)\gamma-(m_{21}-m_\phi)\delta$	$-(m_{12}-m_\phi)\gamma-(m_{22}-m_\phi)\delta$
σ_θ/σ^*	$p_{13}\bar{p}+q_{13}\bar{q}+m_{13}\bar{m}$	$p_{14}\bar{p}+q_{14}\bar{q}+m_{14}\bar{m}$
$(\theta = 0)$	$-(p_{13}-1)\alpha-(p_{23}-1)\beta$	$-(p_{14}-1)\alpha-(p_{24}-1)\beta$
	$-(m_{13}-m_\theta)\gamma-(m_{23}-m_\theta)\delta$	$-(m_{14}-m_\theta)\gamma-(m_{24}-m_\theta)\delta$
σ_ϕ/σ^*	$p_{11}\bar{p}+q_{11}\bar{q}-m_{11}\bar{m}$	$p_{12}\bar{p}+q_{12}\bar{q}-m_{12}\bar{m}$
$(\theta = \pi)$	$-(p_{11}-1)\alpha-(p_{21}-1)\beta$	$-(p_{12}-1)\alpha-(p_{22}-1)\beta$
	$+(m_{11}-m_\phi)\gamma+(m_{21}-m_\phi)\delta$	$+(m_{12}-m_\phi)\gamma+(m_{22}-m_\phi)\delta$
σ_θ/σ^*	$p_{13}\bar{p}+q_{13}\bar{q}-m_{13}\bar{m}$	$p_{14}\bar{p}+q_{14}\bar{q}-m_{14}\bar{m}$
$(\theta = \pi)$	$-(p_{13}-1)\alpha-(p_{23}-1)\beta$	$-(p_{14}-1)\alpha-(p_{24}-1)\beta$
	$+(m_{13}-m_\theta)\gamma+(m_{23}-m_\theta)\delta$	$+(m_{14}-m_\theta)\gamma+(m_{24}-m_\theta)\delta$

The conditions (5) must now be applied at $\theta = 0$ and $\theta = \pi$ as well as to the inner and outer surfaces. Since the pressure, thrust and moment may vary independently between 0 and $\bar{p}$, 0 and $\bar{q}$ and 0 and $\bar{m}$, the problem no longer falls within the category of standard linear programming, but may be made to do so.

The new problem may be stated as follows: two vectors x_j and z_k satisfy a system of inequalities

$$V_i^L \leq [a_{ij}\, a_{ik}^*]\begin{bmatrix} x_j \\ z_k \end{bmatrix} \leq V_i^u \tag{7}$$

It is required to find a vector x_j which satisfies (7) for *all* values of z_k lying within certain limits and which maximizes the function $f = c_j x_j$. The limits on z_k are such that

$$g_k^L \leq z_k \leq g_k^u.$$

Rewrite equation (7) in the form:

$$V_i^L - a_{ik}^* z_k = \phi_i^L(z_k) \leq a_{ij} x_j \leq \phi_i^u(z_k) = V_i^u - a_{ik} z_k$$

This is not equivalent to the standard form since ϕ_i^L and ϕ_i^u are functions of z_k. However, for each value of i the upper and lower limits ϕ_i^u and ϕ_i^L can be found so that:

$$b_i^L = \max\{\phi_i^L(z_k)\} = V_i^L + \max(-a_{ik}^* z_k)$$

$$b_i^u = \min\{\phi_i^u(z_k)\} = V_i^u - \max(a_{ik}^* z_k). \tag{8}$$

The value of $\max(a_{ik}^* z_k)$, for example is determined by noting that if a_{is}^* is positive the corresponding value taken for z_k is g_s^u and if a_{is}^* is negative the corresponding value taken for z_k is g_s^L. In these circumstances equation (8) reduces to the form

$$b_i^L = V_i^L - \sum_k \tfrac{1}{2}\{(1-\text{sign } a_{ik}^*)a_{ik}^* g_k^u + (1+\text{sign } a_{ik}^*)a_{ik}^* g_k^L\}$$

$$b_i^u = V_i^u - \sum_k \tfrac{1}{2}\{(1+\text{sign } a_{ik}^*)a_{ik}^* g_k^u + (1-\text{sign } a_{ik}^*)a_{ik}^* g_k^L\} \tag{9}$$

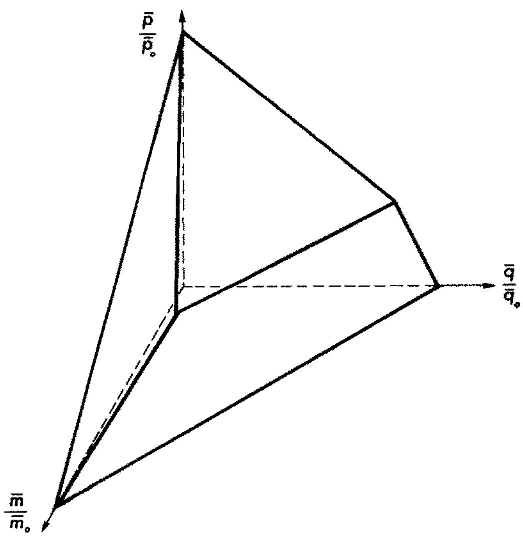

FIG. 8. Interaction surface for pressure, thrust and moment loadings.

The procedure to be adopted is to choose values of $\bar{q}$ and $\bar{m}$, form the limits b_i^L and b_i^u and then proceed according to the standard linear programme to find the corresponding value of $\bar{p}$.

For the geometry $r/R = 0{\cdot}10$, $R/T = 90$, $t/T = 0{\cdot}50$ the interaction surface obtained is shown in Fig. 8.

It is not difficult to show that an interaction surface of the form $\bar{p}/\bar{p}_0 + \bar{q}/\bar{q}_0 + \bar{m}/\bar{m}_0 = 1$ is on the safe side and the result of Fig. 8 supports the use of this simple formula. This formula together with the values of $\bar{p}_0$, $\bar{q}_0$ and $\bar{m}_0$ given in Figs. 4–7 is then a reasonable design method.

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Résumé—Des évaluations de limites inférieures ont été trouvées pour des valeurs de traitement au départ pour des charges de pression, de poussée et de moment appliquées par un bec radial dans un réservoir à pression sphérique; le bec peut être encastré ou saillant. Ces évaluations ont été obtenues en utilisant le théorème de Melan et en tirant parti des solutions d'élasticité déjà disponibles. Les résultats furent obtenus en utilisant des techniques de programmation linéaire standard et ont été présentés sous une forme graphique utile.

Zusammenfassung—Abwärts gerichtete Schätzungen wurden gefunden für die Abnahmewerte von Druck, Schub und Momentbelastung durch eine Querdüse in einem kugelförmigen Druckgefäss; die Düse kann flach oder herausragend sein. Diese Schätzungen wurden durch Verwendung des Melan'schen Thorems und durch Ausnutzung bereits vorhandener elastischer Lösungen erzielt. Normale lineare Programmethoden wurden angewendet und die Resultate werden in nützlicher graphischer Form dargestellt.

Абстракт—Приводятся нижние граничные оценки величин разрушения давлением, ударом и моментными нагрузками, которые приложенны к сосуду высокого давления через радиальную насадку; эта насадка может находится на одном уровне или выступать. Это достигается при использовании теоремы Мелана и исходя из известных уже решений теории упругости. Результаты получаются используя стандардную технику линейного програмирования и представлено в пригодной графической форме.